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THE EVOLUTION OF A TRAPPED MODE OF OSCILLATIONS IN A "STRING ON AN ELASTIC FOUNDATION–MOVING INERTIAL INCLUSION" SYSTEM†

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It is shown that natural vibrations, localized around the inclusion, are possible in a system consisting of an "infinite string on an elastic foundation–concentrated inertial inclusion which moves at a constant, subcritical velocity". The evolution of the trapped mode of oscillations is described analytically for the case of a slowly accelerating inclusion, i.e. the dependence of the amplitude of the oscillations on the frequency is found. The solution which is constructed holds in the time interval when the velocity of the inclusion is not close to the critical velocity (the non-resonance case). An approach to the solutions of similar problems, based on the method of multiple scales, is proposed which enables one to reduce the solution of a problem to investigating an ordinary differential equation with slowly varying coefficients.

The problem considered here of the dynamics of a string on an elastic foundation which is acted upon by a concentrated, moving, inertial load can be of interest for two reasons.

First, it is well known [1] that natural oscillations which are trapped near the inclusion are possible in an "infinite string on an elastic foundation–concentrated, fixed inertial inclusion" system. It is shown below that, if the inertial inclusion moves at a constant subcritical velocity, a trapped modes of oscillations are also possible. If, however, an inclusion is considered which moves along the string with a small acceleration, then the problem of describing the evolution of a trapped mode of oscillations in a system with slowly varying parameters with respect to time naturally arises. The development of an approach to solving such problems is one of the aims of this paper.

Second, it is well known that it is impossible to exceed the critical velocity of an inertia-less, concentrated, accelerating load moving along a string. Solution of the problem in a linear formulation shows that, at the instant when the critical velocity is exceeded, the spatial derivative of the displacements of the string in front of the load becomes infinite [2–4] and, as a consequence, an infinitely large longitudinal force of wave impedance to the motion acts on the load [5]. A consideration of the problem in a non-linear formulation also does not completely resolve this paradox [6]. The hypothesis was put forward in [7] that the paradox can be removed if an inertial load is considered. Then, the force acting on the string from the side of the inclusion at the instant when the critical velocity is exceeded could turn out to be zero when there is a non-zero external force acting on the load. In this case, it would become possible for the critical velocity to be exceeded. Another aim of this paper is to obtain the asymptotic form of the solution of the problem of surmounting the critical velocity of an inertial load in the case of an acceleration which approaches zero. It should be noted that the solution for an accelerating load, which is obtained below, only holds in the time interval when the velocity of the load is not close to the critical velocity (the non-resonance case). Nevertheless, the stage necessary to construct the solution in the resonance case is carried out.

1. FORMULATION OF THE PROBLEM

We introduce the following notation: $u(x, t)$ is the displacement of the point of the string with coordinate x at the instant of time t, T_0 is the tensile strength of the unperturbed string, ρ is the density per unit length of the string, $c = \sqrt{T_0/\rho}$ is the speed of sound (the critical velocity), $l(t)$ is the loading coordinate (a specified function), $\upsilon(t) = \dot{l}(t)$ and $a(t) = \ddot{l}(t)$ are the velocity and acceleration of the load respectively,

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$\beta = \upsilon/c$, $k_0 = T_0 k > 0$ is the coefficient of elasticity of the foundation, $\chi_0(t) = T_0\chi(t)$ is the transverse force acting on the string on the side of the load, $f_0(t) = T_0 f(t)$ is the external transverse force acting on the load, $m_0 = T_0 m$ is the mass of the load, u_0 is the deflection of the string at the point of application of the load and θ is the Heaviside function.

The equations of the dynamics of the string and the load have the form

$$u'' - \frac{1}{c^2}\ddot{u} - ku = -\chi(t)\delta(x - l(t)), \quad m\ddot{u}_0 = -\chi(t) + f(t) \tag{1.1}$$

The kinematic relation

$$u_0(t) = u(l(t), t) \tag{1.2}$$

holds.

Regarding the load, we assume that it suddenly arises when $t = 0$ and, moreover, the string has been in a state of rest up to this instant. We therefore put $\chi(t) \equiv 0$ when $t < 0$. Correspondingly, a generalized Cauchy problem [8] with the conditions

$$u(x,t)|_{t<0} \equiv 0, \quad \dot{u}(x,t)|_{t<0} \equiv 0 \tag{1.3}$$

is set up for the first equation of (1.1).

We shall assume that $l(0) = 0$, $\upsilon(0) = \upsilon_0$ and that $|\upsilon_0| < c$.

We consider the first equation of (1.1) in a system of coordinates $\xi = x - l(t)$, $\tau = t$ which moves together with the load. Taking account of the first equation of (1.1) and relation (1.2), we have

$$(1-\beta^2)u''_{\xi\xi} + \frac{2\upsilon}{c^2}u''_{\xi\tau} + \frac{a}{c^2}u'_{\xi} - \frac{1}{c^2}u''_{\tau\tau} - ku = (mu''_{0\tau\tau} - f)\delta(\xi) \tag{1.4}$$

2. THE INTEGRAL EQUATION FOR THE FUNCTION χ

The first equation of (1.1) is an inhomogeneous Klein–Gordon equation. Its solution can be represented in the form of a convolution of the fundamental solution [8] with the rightful part. Taking (1.3) into account, we obtain [4, 9]

$$u = \frac{c}{2}\int_0^t \chi(\tau)\theta\left(c - \frac{|x - l(\tau)|}{t - \tau}\right) J_0(\Psi(t,\tau,x))d\tau \tag{2.1}$$

$$\Psi(t,\tau,x) = \sqrt{k(c^2(t-\tau)^2 - (x - l(\tau))^2}$$

In the case of subcritical motion of the load, the first equation of (2.1) gives

$$u_0 = \frac{c}{2}\int_0^t \chi(\tau) J_0(\Psi(t,\tau,l(t)))\,d\tau \tag{2.2}$$

On the other hand, by virtue of the second equation of (1.1)

$$u_0 = \frac{1}{m}\int_0^t (f(\tau) - \chi(\tau))(t - \tau)d\tau \tag{2.3}$$

Equating the right-hand sides of Eqs (2.2) and (2.3), we obtain an integral equation of the first kind, the solution of which is $\chi(t)$. Differentiating this equation with respect to t, we obtain the integral equation of the second kind

$$\chi(t) = -\int_0^t \left(\frac{dJ_0(\Psi(t,\tau,l(t)))}{dt} + \frac{2}{mc}\right)\chi(\tau)d\tau + \frac{2}{mc}\int_0^t f(\tau)d\tau \tag{2.4}$$

Integral equation (2.4) is convenient for the numerical solution of the problem.

3. MOTION WITH CONSTANT VELOCITY

We put $a = 0$ and consider the homogeneous ($f = 0$) stationary problem (we drop initial conditions (1.3)) of the natural oscillations of the system described by Eq. (1.4). Suppose

$$u = U(\xi)\exp(-i\Omega\tau), \quad u_0 = U_0\exp(-i\Omega\tau) \tag{3.1}$$

It is well known [1] that, when $\upsilon = 0$, this system has a mixed spectrum of natural oscillations. Sinusoidal travelling waves correspond to the frequencies $|\Omega| \geqslant c\sqrt{k}$ (where $c\sqrt{k}$ is the cutoff frequency). These frequencies form a continuous spectrum. The specific feature of systems with inertial inclusions is the fact that a discrete spectrum of natural frequencies of the oscillations can exist for which $|\Omega| < c\sqrt{k}$ and the modes are localized in the neighbourhood of the inclusions. In such cases, one speaks of *trapped* modes of oscillations. In the given case, a single frequency $\pm\Omega_0$ exists for which the eigenvalue problem being considered has a non-trivial solution; a mode of oscillation which decays exponentially at infinity corresponds to this frequency.

We will show that an analogous situation also holds in the case when $|\upsilon| = \text{const} < c$. Since trapped modes of oscillation are of interest, we will formulate the boundary conditions at infinity in the form

$$u \to 0, \text{ when } \xi \to \pm\infty \tag{3.2}$$

Substituting expressions (3.1) into Eq. (1.4), we obtain

$$U_{\xi\xi} - 2iB(\Omega)U_\xi - A^2(\Omega)U = -\frac{m\Omega^2}{1-\beta^2}U_0\delta(\xi) \tag{3.3}$$

where

$$A^2(\Omega) = \frac{kc^2 - \Omega^2}{c^2 - \upsilon^2}, \quad B(\Omega) = \frac{\upsilon\,\Omega}{c^2 - \upsilon^2} \tag{3.4}$$

The dispersion relation for the operator of the left-hand side of Eq. (3.3) has the form

$$\omega^2 - 2B(\Omega)\omega + A^2(\Omega) = 0 \tag{3.5}$$

whence

$$\omega = B(\Omega) \pm iS(\Omega) \tag{3.6}$$

The solution of Eq. (3.3) when $\upsilon < c$ can be written in the form

$$U(\xi) = U_0\frac{m\Omega_0^2\exp(-S(\Omega_0)|\xi| + iB(\Omega_0)\xi)}{2(1-\beta^2)S(\Omega_0)} \tag{3.7}$$

where

$$S^2(\Omega) = A^2(\Omega) - B^2(\Omega) = \frac{c^2(\Omega_*^2 - \Omega^2)}{(c^2 - \upsilon^2)^2}, \quad \Omega_* = \sqrt{k(c^2 - \upsilon^2)} \tag{3.8}$$

Here, Ω_* is the cutoff frequency for an inclusion moving along the string and $S(\Omega) > 0$ when $\Omega < \Omega_*$.

Noting that $U(0) = U_0$, we obtain the frequency equation

$$m\Omega_0^2 = 2(1-\beta^2)S(\Omega_0) \tag{3.9}$$

from which we find the value of the frequency corresponding to the trapped mode of oscillations

$$\Omega_0^2 = 2m^{-2}c^{-2}\left(\sqrt{1 + m^2c^2k(c^2 - \upsilon^2)} - 1\right) \tag{3.10}$$

On comparing the linear combination of solutions corresponding to the frequencies $\pm\Omega_0$, we obtain an expression for the real solution of Eq. (1.4), which corresponds to oscillations which are trapped in the neighbourhood of the load

$$u = C\exp(-S(\Omega_0)|\xi|)\cos(\Omega_0\tau - B(\Omega_0)\xi + D) \tag{3.11}$$

where C and D are arbitrary constants.

Remark 1. When $\upsilon \to c - 0$, we have

$$\Omega_0^2 = \Omega_*^2 - \frac{k^2m^2c^2}{4}(c^2 - \upsilon^2)^2 + o((c^2 - \upsilon^2)^2)$$
$$S^2(\Omega_0) = \frac{k^2m^2c^4}{4}, \quad |B(\Omega_0)| \sim \sqrt{\frac{k}{1-\beta^2}} \to \infty \tag{3.12}$$

We will now consider the inhomogeneous problem, assuming that the external load f is constant. Applying a generalized Fourier transformation with respect to time to Eq. (1.4) when $a = 0$, we obtain, taking account of the zero initial conditions

$$F[u](\xi,\Omega) = \frac{(m\Omega^2F[u_0](\Omega) + fF[\theta])\exp(-S(\Omega)|\xi| + iB(\Omega)\xi)}{2(1-\beta^2)S(\Omega)} \tag{3.13}$$

where F is the Fourier transformation operator. Hence

$$F[u_0] = \frac{fF[\theta]}{2(1-\beta^2)S(\Omega) - m\Omega^2} \tag{3.14}$$

In the case of large τ, an asymptotic estimate for U_0 can be obtained, starting out from relation (3.14), by the stationary-phase method. It is well known [10] that the motion of the inclusion when $\tau \to \infty$ has the form

$$u_0(\tau) = \sum_{\Omega\in\sigma_0} U_0^{(\Omega)}(\tau)\exp(-i\Omega\tau) + O(\tau^{-5/2}) \tag{3.15}$$

where the summation is carried out over the set of frequencies $\sigma_0 = \{0, \pm\Omega_0, \pm\Omega_*\}$ (which are different when $|\upsilon| < c$). The expressions

$$U_0^{(0)} = \frac{f}{2\sqrt{k(1-\beta^2)}}, \quad 2U_0^{(\pm\Omega_0)} = -\frac{mc^2f}{m^2c^2\Omega_0^2 + 2}, \quad U_0^{(\pm\Omega_*)} = O(\tau^{-3/2}) \tag{3.16}$$

can be obtained for the "amplitudes" $U_0^{(\Omega)}$.

Hence, the solution u_0 is asymptotically represented by the sum of the non-decreasing contributions from the frequency of the trapped made of oscillations Ω_0 and the zero oscillation. The contribution to the solution from the continuous spectrum is asymptotically determined by the decreasing contribution from the cutoff frequency Ω_*.

4. MOTION WITH A SMALL ACCELERATION

We shall assume below that the acceleration $a = O(\varepsilon)$ of the load is small and seek an asymptotic solution of the problem when

$$\tau \to \infty, \quad |\upsilon| = \left|\int_0^\tau a(t)dt\right| < c \tag{4.1}$$

Problems of an inertia-free load on a string [4] and of a Timoshenko beam [11] on a deformable foundation have been considered previously in an analogous formulation. A solution was obtained for the wave fronts using estimates of the convolution integral of Green's function of the Klein–Gordon operator with the right-hand side of the first equation of (1.1).

If the acceleration a of the load is small, the coefficients on the left-hand side of Eq. (1.4) are slowly varying functions of time. The idea behind the approach proposed below is as follows. For long times, the system (when there is no external force) essentially behaves as a one-degree-of-freedom system, and it is therefore natural to expect that the evolution of a trapped mode of oscillations can be described using an ordinary second-order differential equation with slowly varying coefficients. It is preferable to obtain this equation without determining the evolution of the solution of Eq. (1.4) with respect to the spatial variable ξ since the latter problem is exceedingly complex. It turns out that this is possible.

We will seek particular solutions of Eq. (1.4) for $\xi < 0$ and $\xi > 0$ in the form

$$u_\gamma(\xi,\tau) = W_\gamma(X,T)\exp(\varphi_\gamma(\xi,\tau)), \quad X = \varepsilon\chi, \quad T = \varepsilon\tau \tag{4.2}$$

$$\varphi'_\xi = i\omega(X,T), \quad \varphi'_\tau = -i\Omega(X,T)$$

Here, $W(X, T)$, $\omega(X, T)$ and $\Omega(X, T)$ are unknown functions which are to be determined using Eq. (1.4); $W \equiv W_\gamma$ and $\varphi \equiv \varphi_\gamma$ if sign $\xi = \gamma$. We require that the frequencies $\omega(X, T)$ and $\Omega(X, T)$ should satisfy dispersion relation (3.5) and the equation

$$\Omega'_X + \omega'_T = 0 \tag{4.3}$$

which follows from the last two relations of (4.2). The phase $\varphi(\xi, \tau)$ is determined using the formula

$$\varphi = i\int(\omega d\xi - \Omega d\tau) \tag{4.4}$$

The solution of Eq. (1.4) is continuous when $\xi = 0$, and we therefore require the continuity of $W(X, T)$ and $\varphi(\xi, \tau)$ at a point under the load. For u_0, we have

$$u_0(\tau) = W(0, T)\exp(\varphi_0(\tau)), \; \varphi_0(\tau) = \varphi(0, \tau) \tag{4.5}$$

We shall only consider the non-resonance case. This case holds if the quantity $|c - \upsilon(T)|$ is non-zero and all the frequencies from the set, over which the summation on the right-hand side of equality (3.15) is carried out, can be assumed to be different.

We will assume that the characteristic time, after which the velocity υ undergoes substantial changes, is significantly greater than the characteristic time after which the establishment of wave processes takes place in the case when $a = 0$ (so that the asymptotic form of (3.15) describes the solution well). In this case, the solution of the problem of the motion of a load with constant velocity shows that it is natural to make certain "a priori" assumptions regarding the structure of the solution in the case of a small acceleration a. We shall assume that, for long times, the solution of the problem has the form

$$u_0(\tau) = \sum_{\Omega\in\sigma(T)} U_0^{(\Omega)}(T)\exp(\varphi_0^{(\Omega)}(\tau)) \equiv \sum_{\Omega\in\sigma(T)} u_0^{(\Omega)} \tag{4.6}$$

$$\frac{d\varphi_0^{(\Omega)}}{d\tau} = -i\Omega(T)$$

where the summation is carried out over the set of frequencies $\sigma(T) = \{0, \pm\Omega_0(T)\}$ (the frequency Ω_0 depends on the slow time T in accordance with formula (3.10)). In fact, the solution will also obviously contain modes of oscillation with other frequencies; in particular, taking account of the mode with frequency $\Omega_*(T)$ can be important, especially in the resonance case.

The problem of finding a particular solution of Eq. (1.4) which satisfies condition (4.6) therefore arises. In the non-resonance case, the modes of oscillation, which correspond to frequencies from the set σ, can be considered as independent and the contribution from each of these frequencies can be determined independently. The required particular solution can therefore be sought in the form of the superposition of three solutions $u^{(\Omega)}$ which, when sign $\xi = \gamma$, have the form of (4.2), for which $\Omega(0, T) \in \sigma(T)$.

Formally integrating Eq. (1.4) over an infinitesimal neighbourhood of the point $\xi = 0$, we find that solution (4.6) must obey the equation

$$mu''_{0\tau\tau} - f = (1-\beta^2)[u'_\xi] \tag{4.7}$$

Henceforth, the jump in a quantity when $\varepsilon = 0$ is indicated by square brackets.

Equation (4.7) can be split into three equations for each of the functions $u^{(\Omega)}$ by using the linear independence of the terms corresponding to the different frequencies from the set σ.

An approach, based on the method of multiple scales [12], is proposed below which enables us to determine the unknown functions $U_0^{(\Omega)}(T)$.

5. THE CONTRIBUTION FROM THE NATURAL FREQUENCY $\pm\Omega_0$

Suppose $\Omega = \delta\Omega_0$, $\delta = \pm 1$, $\Omega_0 > 0$. Furthermore, for brevity, we shall omit the subscript $(\delta\Omega_0)$ in the case of the functions u_0, ω and φ_0. For the mode of oscillations with frequency $\delta\Omega_0$, we have, by virtue of relations (4.2),

$$mu''_{0\tau\tau} = (1-\beta^2)(u_0[i\omega(0,T)] + \varepsilon[W'_X]\exp(\varphi_0(\tau))) \tag{5.1}$$

The quantity $\omega(0, T) = \omega_\gamma$ is calculated using formula (3.6) when $\Omega = \delta\Omega_0$ (the root which corresponds to a decreasing solution when $\xi \to \infty$ is chosen).

We will now make use of a modification of the method of multiple scales for linear equations with slowly varying coefficients [12, Ch. 7] and seek the function U_0 in the form of the asymptotic series

$$U_0(T) = \sum_{j=0}^{\infty} \varepsilon^j U_0^j(T) \tag{5.2}$$

As is customary when using the method of multiple scales, we will assume that the variables φ_0 and T are independent. Derivatives with respect to τ are transformed according to the rules

$$\begin{aligned} (\cdot)'_\tau &= -i\delta\Omega_0\partial_{\varphi 0} + \varepsilon\partial_T \\ (\cdot)'_{\tau\tau} &= -\Omega_0^2\partial^2_{\varphi 0\varphi 0} - 2\varepsilon i\delta\Omega_0\partial^2_{\varphi 0 T} - \varepsilon i\delta\Omega'_{0T}\partial_{\varphi 0} + \varepsilon^2\partial^2_{TT} \end{aligned} \tag{5.3}$$

and, when these rules and equality (5.1) are taken into account, it can be verified that the zeroth approximation equation is identically satisfied by virtue of frequency equation (3.9). In order to determine the unknown function $U_0^0(T)$, it is necessary to consider the equation of the first approximation

$$2\Omega_0 U^{0\prime}_{0T} + \Omega'_{0T}U_0^0 = im^{-1}\delta(1-\beta^2)[W'_X] \tag{5.4}$$

in which, by virtue of frequency equation (3.9), there is no term associated with the first approximation U_0^1. Equation (5.4) contains the unknown quantity $[W'_X]$. We determine this quantity by considering Eq. (1.4) when $\xi = \pm 0$.

We again use the method of multiple scales. For brevity below, we shall omit the subscript γ in the case of the functions W and ω. We employ an approach which is similar to that used above in the case of Eq. (5.1) and has been extended to the case of a partial differential equation with slowly varying coefficients. We put

$$W(X,T) = \sum_{j=0}^{\infty} \varepsilon^j W^j(X,T) \tag{5.5}$$

The variables φ, X and T are assumed to be independent. The time derivatives in (1.4) are transformed using formulae (5.3), where φ has to be taken instead of φ_0. Furthermore, we have

$$\begin{aligned} \partial_\xi &= i\omega\partial_\varphi + \varepsilon\partial_X \\ \partial^2_{\xi\xi} &= -\omega^2\partial^2_{\varphi\varphi} + 2\varepsilon i\omega\partial^2_{\varphi X} + \varepsilon i\omega'_X\partial_\varphi + \varepsilon^2\partial^2_{XX} \\ \partial^2_{\xi\tau} &= \omega\Omega_0\partial_{\varphi\varphi} + \varepsilon i\omega\partial^2_{\varphi T} - \varepsilon i\delta\Omega_0\partial^2_{\varphi X} + \varepsilon i\omega'_T\partial_\varphi + \varepsilon^2\partial^2_{XT} \end{aligned} \tag{5.6}$$

Substituting the expressions for the derivatives and the asymptotic expansion into Eq. (1.4), we obtain that the zeroth approximation equations are satisfied identically by virtue of dispersion relation (3.5).

In order to determine the first term W^0 of series (5.5), it is necessary to consider the first-approximation equation

$$(1-\beta^2)(2\omega W_X^{0'}+\omega_X' W^0)+2\upsilon c^{-2}(\omega W_T^{0'}-\delta\Omega_0 W_X^{0'}+\omega_T' W^0)+$$
$$+ac^{-2}\omega W^0+c^{-2}\delta(2\Omega_0 W_T^{0'}+\Omega_{0T}' W^0)=0 \tag{5.7}$$

We have

$$\omega_X' = \omega_\Omega' \Omega_X' = -\omega_\Omega' \omega_T' \tag{5.8}$$

by virtue of Eq. (4.3). Transforming Eq. (5.7), we obtain

$$W_X^{0'} = -\frac{1}{2i\gamma S(\delta\Omega_0)(c^2-\upsilon^2)}\sum_{j=0}^{2} F_j \tag{5.9}$$

$$F_0 = a\omega W^0,\quad F_1 = 2(\upsilon\omega+\delta\Omega_0)W_T^{0'}$$

$$F_2 = (-(c^2-\upsilon^2)\omega_\Omega'\omega_T' + 2\upsilon\omega_T' + \delta\Omega_{0T}')W^0$$

Calculating the jump when $X = 0$ from here, we find, on taking account of the fact that $W^0(0, T) = U_0^0(T)$,

$$[W_X^{0'}] = -\frac{(\Phi_0+\Phi_2)U_0^0 + \Phi_1 U_{0T}^{0'}}{iS(\delta\Omega_0)(c^2-\upsilon^2)} \tag{5.10}$$

$$\Phi_0 = aB(\delta\Omega_0),\quad \Phi_1 = 2(\upsilon B(\delta\Omega_0)+\delta\Omega_0)$$

$$\Phi_2 = -(c^2-\upsilon^2)(B_\Omega'(\delta\Omega_0)B_T'(\delta\Omega_0) - S_\Omega'(\delta\Omega_0)S_T'(\delta\Omega_0)) + 2\upsilon B_T'(\delta\Omega_0) + \delta\Omega_{0T}' = \Phi_3 + \Phi_4$$

$$\Phi_3 = (c^2-\upsilon^2)S_\Omega'(\delta\Omega_0)S_T'(\delta\Omega_0),\quad \Phi_4 = \upsilon B_T'(\delta\Omega_0) + \delta\Omega_{0T}'$$

Substituting the first equation of (5.10) into Eq. (5.4) and using frequency equation (3.9), we obtain

$$2\Omega_0 U_{0T}^{0'} + \Omega_{0T}' U_0^0 = -\left.\frac{2(1-\beta^2)((\Phi_0+\Phi_2)U_0^0 + \Phi_1 U_{0T}^{0'})}{m^2c^2\Omega_0^2}\right|_{\delta=1} \tag{5.11}$$

Equation (5.11) enables us to determine the unknown function $U_0^0(T)$. The coefficient of $U_{0T}^{0'}$ is equal to

$$2\Omega_0 + \left.\frac{2(1-\beta^2)\Phi_1}{m^2c^2\Omega_0^2}\right|_{\delta=1} = 2\frac{m^2c^2\Omega_0^2+2}{m^2c^2\Omega_0} \tag{5.12}$$

We now use the notation

$$z = m^2c^2\Omega_0^2 + 2,\quad \lambda = 1 + m^2c^2k(c^2-\upsilon^2) = z^2/4 \tag{5.13}$$

The solution of Eq. (5.11) has the form

$$U_0^0 = \frac{C_0}{2}J_1 \exp\left(-\int \frac{(1-\beta^2)(\Phi_0+\Phi_2)|_{\delta=1}}{\Omega_0 z}dT\right) \tag{5.14}$$

$$J_1 = \exp\left(-\int \frac{m^2c^2\Omega_0}{2z}d\Omega_0\right) = z^{-1/4}$$

where C_0 is an arbitrary constant. It remains to evaluate the integral on the right-hand side of the first equation of (5.14). We have

$$S'_T = \frac{m\Omega_0\Omega'_0}{1-\beta^2} + \frac{mc^2 \upsilon a\Omega_0^2}{(c^2-\upsilon^2)^2}, \quad \Phi_4|_{\delta=1} = \frac{a\upsilon\,\Omega_0}{c^2-\upsilon^2} + \frac{\Omega'_0}{1-\beta^2} + \frac{2a\upsilon^3\Omega_0}{(c^2-\upsilon^2)^2}$$

$$J_2 = \exp\left(-\int \frac{(1-\beta^2)\Phi_0|_{\delta=1}}{\Omega_0 z} dT\right)$$

$$J_3 = \exp\left(-\int \frac{(1-\beta^2)\Phi_3|_{\delta=1}}{\Omega_0 z} dT\right) = J_1^2 J_8$$

$$J_8 = \exp\left(-\int \frac{m^2\Omega_0^2 \upsilon}{(1-\beta^2)z} d\upsilon\right) = J_9 J_{10}$$

$$J_9 = \exp\left(-\int \frac{d\upsilon^2}{2(c^2-\upsilon^2)}\right) = \sqrt{c^2-\upsilon^2}$$

$$J_{10} = \exp\left(\int \frac{d\upsilon^2}{2(c^2-\upsilon^2)\sqrt{\lambda}}\right)$$

$$J_4 = \exp\left(-\int \frac{(1-\beta^2)\Phi_4|_{\delta=1}}{\Omega_0 z} dT\right) = J_2 J_5 J_6 = J_2^{-1} J_5 J_{10}^{-1}$$

$$J_5 = \exp\left(-\int \frac{d\Omega_0}{2\Omega_0}\right)\exp\left(\int \frac{m^2c^2\Omega_0 d\Omega_0}{2z}\right) = \frac{1}{J_1\sqrt{\Omega_0}}$$

$$J_6 = \exp\left(\int \frac{d\lambda}{2(\lambda-1)\sqrt{\lambda}}\right)\exp\left(-\frac{1}{m^2c^4k}\int \frac{d\lambda}{2\sqrt{\lambda}}\right) = J_{10}^{-1} J_2^{-2}$$

Finally, we obtain

$$U_0^0 = \frac{C_0}{2}\frac{J_1^2 J_{10}}{\Omega_0^{1/2}} = \frac{C_0}{2}\sqrt{\frac{c^2-\upsilon^2}{\Omega_0(m^2c^2\Omega_0^2+2)}} \tag{5.15}$$

Remark 2. When $\upsilon \to c-0$, we have $U_0^0 = O((c^2-\upsilon^2)^{1/4})$ by virtue of the second expression of (3.8) and the asymptotic formulae (3.12).

6. THE CONTRIBUTION FROM THE FREQUENCY $\Omega = 0$

The existence in the solution of a contribution with zero frequency is due to the fact that a constant external force acts on the system. Taking into account dispersion relation (3.5) and the boundary conditions at infinity (3.2), Eq. (4.7) gives

$$u_0^{(0)} = \frac{f}{2S(0)(1-\beta^2)} = \frac{f}{2\sqrt{k(1-\beta^2)}} \tag{6.1}$$

The right-hand side of this equation is identical with the right-hand side of the first equation of (3.16) and, also, with the subcritical solution of the problem of the slow acceleration of an inertia-free load [4].

To sum up, we have obtained

$$u_0 \simeq C_0\sqrt{\frac{c^2-\upsilon^2}{\Omega_0(m^2c^2\Omega_0^2+2)}}\cos\left(\int_0^t \Omega_0(T)dT\right) + \frac{f}{2\sqrt{k(1-\beta^2)}} \tag{6.2}$$

In order to determine the unknown constant C_0, it is necessary to make use of the initial conditions. We assume that $U_0^{(\Omega_0)}$, when $t=0$, is equal to the expression on the right-hand side of the second equality of (3.16), calculated when $\upsilon = \upsilon_0$.

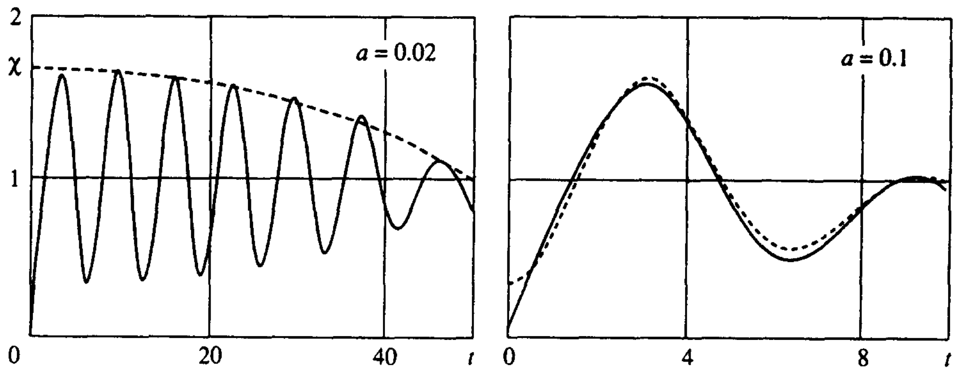

Fig. 1

7. COMPARISON WITH NUMERICAL RESULTS. CONCLUSIONS

Numerical calculations, based on Eq. (2.4), were carried out in order to check the asymptotic solution constructed above. These calculations were carried out using the following values of the parameters: $c = 1, k = 2, m = 2, f = 1, \upsilon_0 = 0$ when $a = 0.02$ and $a = 0.1$ (Fig. 1). The solid curves represent the numerical results and the dashed lines show the analytical results (a graph of the envelope of the analytical solution is given on the left). It is seen that, even if the acceleration of the load is quite large, such that the time interval from the beginning of the acceleration up to the instant when the critical velocity is exceeded fits into just one and a half periods of the natural oscillations of the inclusion, the asymptotic solution constructed is in good agreement with the calculated solution. As would be expected, this solution inadequately describes the dynamic process at short times and in the case of values of υ which are close to c. Note that the results of the investigation of the problem even in the non-resonance case and, also, the results for the resonance case enable us to assume that, as the velocity approaches the critical velocity, a system with an inertial load behaves in the qualitatively similar manner to a system with an inertia-free load.

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